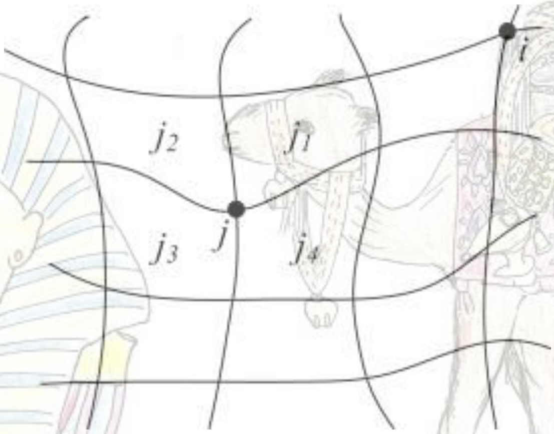
3維邊界元素法四邊形線性元素



相鄰元素

以 n 個四邊形線性元素依 2 維元素所述方法，對邊界積分方程式加以離散如下。

$$\frac{1}{2}\phi_i+\sum_{j=1}^{n}\phi_j\int_{A_j}\bar{\phi}^{*}\,dA=\sum_{j=1}^{n}\bar{\phi}_j\int_{A_j}\phi^{*}\,dA$$

對被積分節點 j，其相鄰元素如上圖，依逆時針方向以 j_1、j_2、j_3及 j_4表示，得下列和分方程式

$$\frac{1}{2}\phi_i+\sum_{j=1}^{n}\sum_{s=1}^{4}h_{ij}^{s}\phi_j=\sum_{j=1}^{n}\sum_{s=1}^{4}g_{ij}^{s}\overline{\phi_j} \qquad (i=1,2,\cdots,n) \qquad (A)$$

$$h_{ij}^{s}=\int_{\Gamma_{k_s}}\beta_s\bar{\phi}^{*}dA=-\frac{1}{16\pi}\int_{-1}^{1}\int_{-1}^{1}\beta_s\frac{1}{r^2}\frac{\partial r}{\partial n}|G|_{\Gamma_{j_s}}\,d\xi_1 d\xi_2 \qquad (s=1\sim4)$$

$$g_{ij}^{\;s}=\int_{\Gamma_{k_s}}\beta_s\phi^{*}dA=\frac{1}{16\pi}\int_{-1}^{1}\int_{-1}^{1}\beta_s\phi^{*}\,|G|_{\Gamma_{j_s}}\,d\xi_1 d\xi_2 \qquad (s=1\sim4)$$

$$r=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}$$

$$\frac{\partial r}{\partial n}=\frac{\partial r}{\partial x}\frac{\partial x}{\partial n}+\frac{\partial r}{\partial y}\frac{\partial y}{\partial n}+\frac{\partial r}{\partial z}\frac{\partial z}{\partial n}$$

$$\left.\begin{aligned}
\beta_1 &= \frac{1}{4}(1-\xi_1)(1-\xi_2)\\
\beta_2 &= \frac{1}{4}(1+\xi_1)(1-\xi_2)\\
\beta_3 &= \frac{1}{4}(1+\xi_1)(1+\xi_2)\\
\beta_4 &= \frac{1}{4}(1-\xi_1)(1+\xi_2)
\end{aligned}\right\}$$

對各被積分元素

$$|G|_{\Gamma_{k_s}} = \sqrt{g_1^{\,2}+g_2^{\,2}+g_3^{\,2}} \qquad (s=1\sim4)$$

$$\left.\begin{aligned}
g_1 &= \frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta}-\frac{\partial z}{\partial \xi}\frac{\partial y}{\partial \eta}\\
g_2 &= \frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta}-\frac{\partial x}{\partial \xi}\frac{\partial z}{\partial \eta}\\
g_3 &= \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}
\end{aligned}\right\}$$

$i \neq j$ 時，應用 Guass 積分進行數值積分得

$$h_{ij}^{s} = -\frac{1}{16\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} w_l w_m \beta_s \frac{1}{r_{ilm}^{\,2}}\frac{\partial r_{ilm}}{\partial n}|G|_{\Gamma_{j_s}} \qquad (s=1\sim4)$$

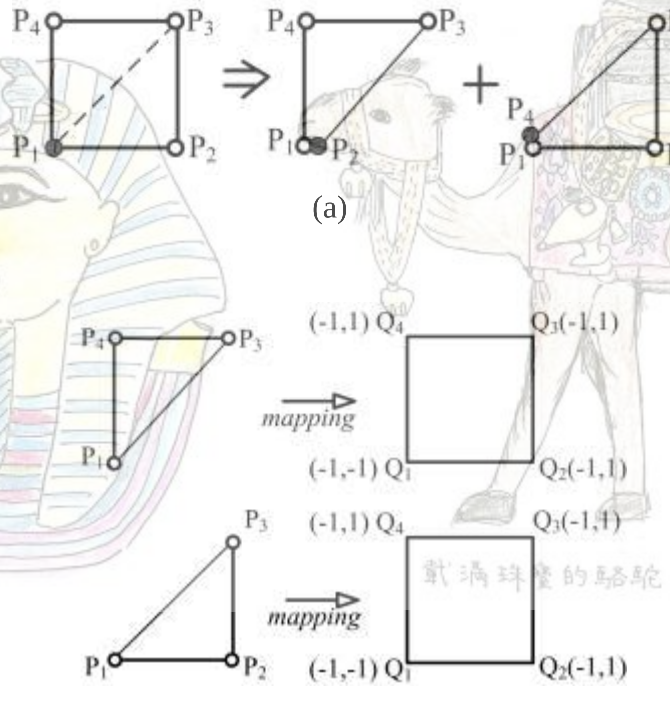
$$g_{ij}^{s} = \frac{1}{16\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} w_l w_m \beta_s \frac{1}{r_{ilm}}|G|_{\Gamma_{j_s}} \qquad (s=1\sim4) \qquad \text{(B)}$$

$$\frac{\partial r_{ilm}}{\partial n} = \frac{x_{lm}-x_i}{r_{ilm}}\left(\frac{\partial x}{\partial n}\right)_j + \frac{y_{lm}-y_i}{r_{ilm}}\left(\frac{\partial y}{\partial n}\right)_j + \frac{z_{lm}-z_i}{r_{ilm}}\left(\frac{\partial z}{\partial n}\right)_j$$

r_{ilm} 為源點 i 至被積分元素(j)的 Guass 積分點(ξ_l，η_m)間距離，w_l 、 w_m 為加權函數，n=2 時，$w_l = w_m = 1$。

$i = j$ 時，由於 $\partial r / \partial n = 0$ 得

$$\hat{h}_{ij} = 0 \qquad (s=1\sim4)$$



(b)

奇異積分處理示意圖

(B)式，$i=j$ 時會產生特異值，必須作下列處理，如上圖(a)，對某被積分元素，節點為 P_1、P_2、P_3 及 P_4，討論節點為 P_1 時，將四邊形元素分割成三角形元素 $\Delta P_1 P_3 P_4$ 及 $\Delta P_1 P_2 P_3$，將 $\Delta P_1 P_3 P_4$ 保角變換成如上圖(b)所示正方形元素(對 $\Delta P_1 P_2 P_3$ 也作同樣處理)，2 者間座標關係如下

$$x = \sum_{k=1}^{4} \beta_k \tilde{x}_k \qquad \text{(C)}$$

$\tilde{x}_k$ (k=1～4)為 Q_1～Q_4 點在實際 3 度空間內的座標，P_k 點座標為 x_k 時

$$\left.\begin{aligned} X_1 &= \tilde{x}_1 = \tilde{x}_2 \\ x_3 &= \tilde{x}_3 \\ x_4 &= \tilde{x}_4 \end{aligned}\right\}$$

將(C)式代入上式得

$$
\begin{aligned}
x &= \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 \\
&= \frac{1}{2}(1-\xi_2)x_1 + \beta_3 x_3 + \beta_4 x_4 \\
&= \sum_{k=1}^{3} \psi_k x_k
\end{aligned}
$$

$$\psi_1 = \frac{1}{2}(1-\xi_2)$$

$$\psi_2 = \beta_3 = \frac{1}{4}(1+\xi_1)\ \ (1+\xi_2)$$

$$\psi^3 = \beta_4 = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

源點座標為 x_1，源點元素內任意 1 點位置向量 r 為

$$
\begin{aligned}
r = x - x_1 &= \sum_{k=1}^{3} \psi_k x_k - x_1 \\
&= \frac{1}{2}(1+\xi_2)\left(\beta_1^* x_1 + \beta_3^* x_3 + \beta_4^* x_4\right) \\
&= \rho\, r^*
\end{aligned}
$$

$$\rho = \frac{(1+\xi_2)}{2}$$

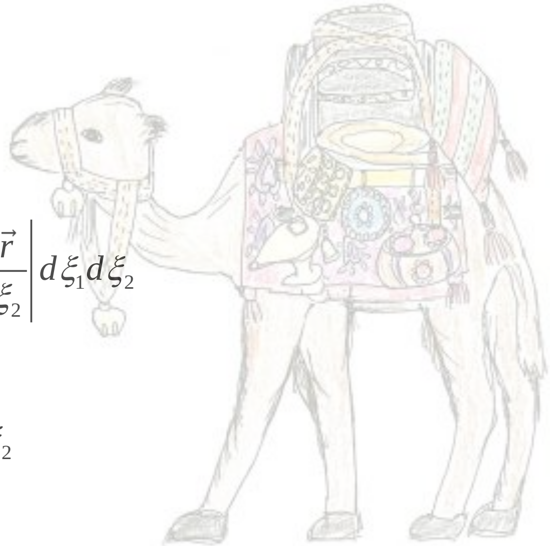
$$r^* = \beta_1^* x_1 + \beta_3^* x_3 + \beta_4^* x_4$$

$$\left.\begin{aligned}
\beta_1^* &= -1 \\
\beta_3^* &= \frac{1}{2}(1+\xi_1) \\
\beta_4^* &= \frac{1}{2}(1-\xi_1)
\end{aligned}\right\}$$

得源點與元素內任意一點間距離 r

$$r = |\rho|\, r^*$$

$r^* = \sqrt{r_1^{*2} + r_2^{*2} + r_3^{*2}}$, r_1^* 、 r_2^* 及 r_3^* 各為 r^* 在 x、y 及 z 方向分量。
$x \to x_1$ 時，$|\rho| \to 0$ 但 $r^* \neq 0$ ，因此對 ξ_1 ，ξ_2 座標的平面元素 $d\Gamma$ ，得



載滿珠寶的駱駝

$$d\Gamma = \left| \frac{\partial \vec{r}}{\partial \xi_1} \times \frac{\partial \vec{r}}{\partial \xi_2} \right| d\xi_1 d\xi_2$$

$$= |\rho| \left| \left(\sum_{k=3}^{4} \frac{\partial \beta_k^*}{\partial \xi_1} x_k \right) \times \frac{\partial \vec{r}}{\partial \xi_2} \right| d\xi_1 d\xi_2$$

$$= |\rho| \left| \frac{\partial \vec{r}_*}{\partial \xi_1} \times \frac{\partial \vec{r}}{\partial \xi_2} \right| d\xi_1 d\xi_2$$

即

$$\frac{1}{r} d\Gamma = \frac{1}{r^*} \left| \frac{\partial \vec{r}^*}{\partial \xi_1} \times \frac{\partial \vec{r}}{\partial \xi_2} \right| d\xi_1 d\xi_2$$

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$$\left. \begin{aligned} \frac{\partial x^*}{\partial \xi} &= \frac{1}{2}(x_3 - x_4) \\ \frac{\partial y^*}{\partial \xi} &= \frac{1}{2}(y_3 - y_4) \\ \frac{\partial z^*}{\partial \xi} &= \frac{1}{2}(z_3 - z_4) \end{aligned} \right\} \qquad \text{(D)}$$

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由上式可知特異性已被消除。
同理，源點為 P_2 時得

$$r^* = \beta_2^* x_2 + \beta_3^* x_3 + \beta_4^* x_4$$

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$$\left. \begin{aligned} \beta_2^* &= -1 \\ \beta_3^* &= \frac{1}{2}(1+\xi_1) \\ \beta_4^* &= \frac{1}{2}(1-\xi_1) \end{aligned} \right\}$$

$\dfrac{\partial x^*}{\partial \xi_1}$、$\dfrac{\partial y^*}{\partial \xi_1}$、$\dfrac{\partial z^*}{\partial \xi_1}$ 值如(D)式所示。

源點為 P_3 時得

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$$r^* = \beta_1^* x_1 + \beta_2^* x_2 + \beta_3^* x_3$$

$$\left.\begin{aligned} \beta_1^* &= \frac{1}{2}(1-\xi_1) \\ \beta_2^* &= \frac{1}{2}(1+\xi_1) \\ \beta_3^* &= -1 \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{\partial x^*}{\partial \xi_1} &= \frac{1}{2}(x_2 - x_1) \\ \frac{\partial y^*}{\partial \xi_1} &= \frac{1}{2}(y_2 - y_1) \\ \frac{\partial z^*}{\partial \xi} &= \frac{1}{2}(z_2 - z_1) \end{aligned}\right\} \qquad \text{(E)}$$

源點為 P_4 時得

$$r^* = \beta_1^* x_1 + \beta_2^* x_2 + \beta_4^* x_4$$

$$\left.\begin{aligned} \beta_1^* &= \frac{1}{2}(1-\xi_1) \\ \beta_2^* &= \frac{1}{2}(1+\xi_1) \\ \beta_4^* &= -1 \end{aligned}\right\}$$

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$\dfrac{\partial x^*}{\partial \xi_1}$、$\dfrac{\partial y^*}{\partial \xi_1}$、$\dfrac{\partial z^*}{\partial \xi_1}$ 值如(E)式所示，因此得

$$g_{ij}{}^{1}=\frac{1}{16\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1-\xi_l-\eta_m+\xi_l\eta_m\right)\frac{1}{r^{*}{}_{ilm}}\left|G^{*}\right|_{\Gamma_{j1}}$$

$$g_{ij}{}^{2}=\frac{1}{16\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1+\xi_l-\eta_m-\xi_l\eta_m\right)\frac{1}{r^{*}{}_{ilm}}\left|G^{*}\right|_{\Gamma_{j2}}$$

$$g_{ij}{}^{3}=\frac{1}{16\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1+\xi_l+\eta_m+\xi_l\eta_m\right)\frac{1}{r^{*}{}_{ilm}}\left|G^{*}\right|_{\Gamma_{j3}}$$

$$g_{ij}{}^{4}=\frac{1}{16\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1-\xi_l+\eta_m-\xi_l\eta_m\right)\frac{1}{r^{*}{}_{ilm}}\left|G^{*}\right|_{\Gamma_{j4}}$$

$$\left|G^{*}\right|_{\Gamma_{j_s}}=\sqrt{g_1^{*2}+g_2^{*2}+g_3^{*2}}\qquad (s=1\sim4)$$

$$\left.\begin{aligned}
g_1{}^{*}&=\frac{\partial y^{*}}{\partial\xi_1}\frac{\partial z}{\partial\xi_2}-\frac{\partial z^{*}}{\partial\xi_1}\frac{\partial y}{\partial\xi_2}\\
g_2{}^{*}&=\frac{\partial z^{*}}{\partial\xi_1}\frac{\partial x}{\partial\xi_2}-\frac{\partial x^{*}}{\partial\xi_1}\frac{\partial z}{\partial\xi_2}\\
g_3{}^{*}&=\frac{\partial x^{*}}{\partial\xi_1}\frac{\partial y}{\partial\xi_2}-\frac{\partial y^{*}}{\partial\xi_1}\frac{\partial x}{\partial\xi_2}
\end{aligned}\right\}$$

$r^{*}{}_{ilm}$ 為源點 i 至被積分元素(j)的 Guass 積分點(ξ_l, η_m)間距離。

$$\left.\begin{aligned}
\frac{\partial x}{\partial\xi_1}&=\frac{1}{4}\left[-\left(1-\xi_2\right)x_1+\left(1-\xi_2\right)x_2+\left(1+\xi_2\right)x_3-\left(1+\xi_2\right)x_4\right]\\
\frac{\partial x}{\partial\xi_2}&=\frac{1}{4}\left[-\left(1-\xi_1\right)x_1-\left(1+\xi_1\right)x_2+\left(1+\xi_1\right)x_3+\left(1-\xi_1\right)x_4\right]
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{\partial y}{\partial\xi_1}&=\frac{1}{4}\left[-\left(1-\xi_2\right)y_1+\left(1-\xi_2\right)y_2+\left(1+\xi_2\right)y_3-\left(1+\xi_2\right)y_4\right]\\
\frac{\partial y}{\partial\xi_2}&=\frac{1}{4}\left[-\left(1-\xi\right)y_1-\left(1+\xi\right)y_2+\left(1+\xi\right)y_3+\left(1-\xi\right)y_4\right]
\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial z}{\partial \xi_1}&=\frac{1}{4}\left[-(1-\xi_2)z_1+(1-\xi_2)z_2+(1+\xi_2)z_3-(1+\xi_2)z_4\right]\\ \frac{\partial z}{\partial \xi_2}&=\frac{1}{4}\left[-(1-\xi_1)z_1-(1+\xi_1)z_2+(1+\xi_1)z_3+(1-\xi_1)z_4\right]\end{aligned}\right\}$$

依上述數值計算，將(A)式以下列矩陣形式表示

$$\Phi = K\overline{\Phi}$$

即表示在邊界表面ϕ與$\overline{\phi}$間的關係式。

$$K = H^{-1}G$$

$$H = H_{ij} = \begin{cases}\hat{H}_{ij} & i \neq j\\ \hat{H}_{ij}+\frac{1}{2} & i = j\end{cases}$$

$$G = G_{ij}$$

$$\left.\begin{aligned}\left[\hat{H}\right]&=\sum_{s=1}^{4} h_{ij}^{s}\\ \left[G\right]&=\sum_{s=1}^{4} g_{ij}^{s}\end{aligned}\right\}$$

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