Navier-Stokes運動方程式

對領域Ω，動量成份若以下式表示

$$\int_\Omega \rho v_1 d\Omega,\quad \int_\Omega \rho v_2 d\Omega$$

則動量分量的變化率為

$$\int_\Omega \frac{\partial \rho v_1}{\partial t} d\Omega,\quad \int_\Omega \frac{\partial \rho v_2}{\partial t} d\Omega$$

從邊界Γ流入各動量分量的流入率為

$$-\int_\Gamma \rho v\cdot v_1 d\Gamma,\quad -\int_\Gamma \rho v\cdot v_2 d\Gamma$$

邊界Γ上的應力度 p_1、p_2 與內部物體力 b_1、b_2 所引起的力矩為

$$\int p_1 d\Gamma + \int b_1 d\Omega,\quad \int p_2 d\Gamma + \int b_2 d\Omega$$

兩者相等時，得下式所示動量守恆方程式

$$\int_\Omega \frac{\partial \rho v_1}{\partial t} d\Omega = -\int \rho v \cdot v_1 d\Gamma + \int p_1 d\Gamma + \int b_1 d\Omega \qquad (1)$$

$$\int \frac{\partial \rho v_2}{\partial t} d\Omega = -\int \rho v \cdot v_2 d\Gamma + \int p_2 d\Gamma + \int b_2 d\Omega \qquad (2)$$

當表面力與應力度間以下列關係式表示

$$p_1 = \sigma_{11} n_1 + \sigma_{12} n_2$$

$$p_2 = \sigma_{21} n_1 + \sigma_{22} n_2$$

邊界Γ上的流速v，若依其成分及方向餘弦(n_1, n_2)以下列形式表示，

$$v = v_1 n_1 + v_2 n_2$$

利用發散定理將上兩式右邊的邊界積分轉換成領域積分，可將(1)、(2)式改寫成

$$\int_\Omega \frac{\partial \rho v_1}{\partial t} d\Omega = -\int \left[\frac{\partial}{\partial x_1}\left(\rho v_1^2\right) + \frac{\partial}{\partial x_2}\left(\rho v_1 v_2\right) \right] d\Omega + \int \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} \right) d\Omega + \int b_1 d\Omega$$

$$\int_\Omega \frac{\partial \rho v_2}{\partial t} d\Omega = -\int \left[\frac{\partial}{\partial x_1}\left(\rho v_1 v_2\right) + \frac{\partial}{\partial x_2}\left(\rho v_2^2\right) \right] d\Omega + \int \left(\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} \right) d\Omega + \int b_2 d\Omega$$

考量質量守恆，得下列運動方程式

$$\rho \left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} \right) = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + b_1 \qquad (3)$$

$$\rho \left(\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} \right) = \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + b_2 \qquad (4)$$

流體粘性係數為 μ 時，應力度可以下式表示

$$\sigma_{ij} = -P\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\varepsilon = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

P 為壓力，Kronecker δ_{ij} 為

$$\begin{cases} i = j & \delta_{ij} = 1 \\ i \neq j & \delta_{ij} = 0 \end{cases}$$

(3)、(4)式為表示非壓縮性粘性流體的基本方程式，稱為 Navier-Stokes 運動方程式。

若將物體力 b 與應力度 σ，以下式無因次化

$$f_i = b_i \frac{L^2}{\mu U} \qquad (5)$$

$$\tau_{ij} = \sigma_{ij} \frac{L}{\mu U}$$

可得下列無次因 Navier-Stokes 運動方程式

$$\mathrm{Re}\left[\frac{\partial u_1}{\partial \tau} + u_1 \frac{\partial u_1}{\partial X_1} + u_2 \frac{\partial u_1}{\partial X_2}\right] = \frac{\partial \tau_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + f_1$$

$$\mathrm{Re}\left[\frac{\partial u_2}{\partial \tau} + u_1 \frac{\partial u_2}{\partial X_1} + u_2 \frac{\partial u_2}{\partial X_2}\right] = \frac{\partial \tau_{21}}{\partial X_1} + \frac{\partial \tau_{22}}{\partial X_2} + f_2$$

Re 為下式所示 Reynolds 數

$$\mathrm{Re} = \frac{\rho L U}{\mu} = \frac{LU}{\nu}$$

應力度、壓力及流速間的關係如下

$$\tau_{ij} = -\mathrm{Re}\cdot p\delta_{ij} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}$$

但壓力 p 作下列無次因化

$$p = P \frac{1}{\rho U^2}$$

在熱對流問題，若只考量因溫度引起密度產生變化時，物體力可以下式表示

$$b_1 = 0$$
$$b_2 = \rho g \beta (T - T_0) \qquad (6)$$

g 為重力加速度、β 為體膨脹率、$T - T_0$ 為溫度差、T_0 為流體靜止時的溫度。

將(6)式代入(5)式可得

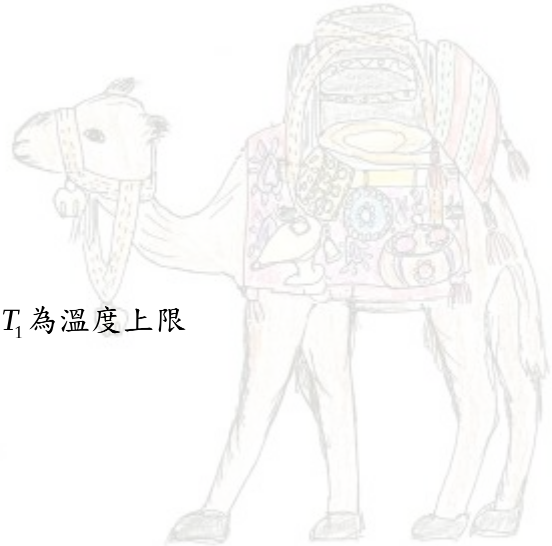
$$f_2 = \frac{Gr}{\mathrm{Re}}\theta$$

Gr 為下式所示 Grashof 數

$$Gr = \frac{g\beta\left(T - T_0\right)L^3}{\nu^2}$$

θ 為無因次溫度，T_0 為溫度下限，T_1 為溫度上限

$$\theta = \frac{T - T_0}{T_1 - T_0}$$



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